Trajectories most slowly going away

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RESUMEN

The controlled object $x' = Ax + u$, $u \in U$, of the second order with complex eigenvalues is considered. The real part of the eigenvalues are assumed to be positive. The control region U is a convex polygon with the origin in its interior. The the controllability region Σ is an open, convex, bounded set in the state plane R^2.

In the article it is proved that each tajectory $x(t)$ starting from a point $x_0 \notin \Sigma$ and satisfying the maximum principle is in a sense, the most slowly going away trajectory. This idea is formulated exactly with the help of Bellman function and justified. This explains the meaning of the maximum principle outside of the controllability region.

I. Introduction

Consider a linear controlled object of the second order

$$x' = Ax + u, \qquad u \in U, \tag{1}$$

where

$$A = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

is a matrix with constant elements; $x = (x^1, x^2)^T \in R^2$ is the state vector of the object (it is contravariant, *i.e.*, a column vector), and the control region U is a convex polygon containing the origin in its interior. The equation (1) can be written in the coordinate form:

$$x'^i = \sum_{j=1,2} a_j^i x^j + u^i, \qquad i = 1, 2.$$

A control $u(t)$, $t_0 \le t \le t_1$, is admissible, if it is piecewise continuous and $u(t) \in U$ for all t, $t_0 \le t \le t_1$. We always assume that the admissible controls are right continuous, *i.e.*, $u(t) = u(t+0)$ for $t_0 \le t \le t_1$, and moreover, $u(t_1) = u(t_1 - 0)$.

Let $u(t)$, $t_0 \le t \le t_1$ be an admissible control and $x_0 \in R^2$ be a given initial point. The solution $x(t)$, $t_0 \le t \le t_1$, of the equation $x' = Ax + u(t)$ with the initial condition $x(t_0) = x_0$ is said to be the trajectory corresponding to the control $u(t)$ with the initial point x_0. We say that the control $u(t)$, $t_0 \le t \le t_1$, *transits* the initial point x_0 to the terminal state $x(t_1)$. Every trajectory is continuous; moreover, it is differentiable for all t except a finite number of moments t.

We consider the time-optimal problem. To transit the given initial point x_0 to the origin in the shortest time by an admissible control. The control and the trajectory which solve this problem are said to be time-optimal (or *optimal*).

We formulate the fundamental facts of the linear optimal control theory [1, 2] for the considered object (1). Assume that the control region U is situated in the general position with respect to the matrix A. In other words, for every edge $[p, q]$ of the convex polygon U, the vector $q - p$ is not an eigenvector of the matrix A.

We denote by Σ_∞ the controllability region, *i.e.*, the set of all initial points x_0 which can be transited to the origin by an admissible control. The set $\Sigma_\infty \subset R^2$ is open and convex. For every point $x_0 \in \Sigma_\infty$ there is the unique optimal process $u(t), x(t)$, transiting x_0 to the origin. Moreover, the optimal control $u(t)$ is piecewise constant, takes its values only at the vertices of the polygon U, and has a finite number of switchings, *i.e.*, a finite number of intervals of constancy.

To find optimal controls, consider the conjugate equation:

$$\psi' = -\psi A, \tag{2}$$

where $\psi = (\psi_1, \psi_2)$ is an auxiliary covariant vector (*i.e.*, a line vector).

The equation (2) has the coordinate form

$$\psi_j' = -\sum_{i=1,2} \psi_i a_j^i, \qquad j = 1, 2.$$

A control $u(t)$, $t_0 \le t \le t_1$, is said to satisfy the *maximum condition*, if there exists a nontrivial solution $\psi(t)$ of equation (2) such that for the scalar product $\langle \psi(t), u \rangle = \psi_1(t)u^1 + \psi_2(t)u^2$, the relation

$$u(t) \arg\max_{u \in U} \langle \psi(t), u \rangle, \qquad t_0 \le t \le t_1, \tag{3}$$

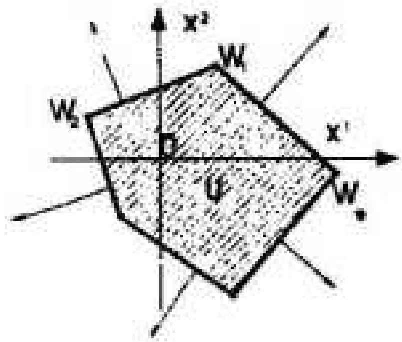


FIGURA 1.

holds, *i.e.*, $\langle\psi(t), u(t)\rangle = \max_{u\in U}\langle\psi(t), u\rangle$. Now the maximum principle affirms: An admissible control $u(t)$, $t_0 \le t \le t_1$, transiting x_0 to the origin, is optimal if and only if it satifies the maximum condition (3) with respect to a nontrivial solution $\psi(t)$ of equation (2).

Let now $X(t)$ be the principal matrix solution of the homogeneous equation

$$x' = Ax, \tag{4}$$

i.e., $X'(t) = AX(t)$ and $X(0) = I$, the identity matrix. Then

$$x(t) = X(t-t_0)\Big(x_0 + \int_{t_0}^{t} X^{-1}(s-t_0)u(s)\,ds\Big), \qquad t_0 \le t \le t_1, \tag{5}$$

is the trajectory corresponding to the control $u(t)$, $t_0 \le t \le t_1$, and the initial point x_0.

In the sequel, we consider the case when the matrix A has two complex eigenvalues λ_1, λ_2 with positive real parts. In this case the general position condition for U is satisfied, since A has no real one-dimensional invariant subspaces. Then the origin is a nonstable singular point (a nonstable focus) for the corresponding homogeneous system (4) and the controllability region Σ_∞ is a bounded, open, convex set in R^2.

II. The Optimal Synthesis

Denote by $w_1, \ldots, w_q$ the vertices of the polygon U, following its counter counterclockwise (figure 1). First we consider the case when the matrix A has the special form:

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with positive a, b, *i.e.*, we consider the controlled object

$$
\begin{aligned}
x^{1'} &= ax^1 - bx^2 + u^1, \\
x^{2'} &= bx^1 + ax^2 + u^2.
\end{aligned}
\tag{6}
$$

In this case the eigenvalues of the matrix A are $\lambda_1 = a+bi$ and $\lambda_2 = a-bi$. The general solution $x(t)$ of the corresponding homogeneous system

$$
\begin{aligned}
x^{1'} &= ax^1 - bx^2, \\
x^{2'} &= bx^1 + ax^2,
\end{aligned}
\tag{7}
$$

has the form:

$$
\begin{aligned}
x^1(t) &= ce^{at}\cos(bt+\theta), \\
x^2(t) &= ce^{at}\sin(bt+\theta),
\end{aligned}
$$

$c > 0$ and θ being constant parameters. Passing to the polar coordinates, we rewrite this general solution in the form:

$$
\begin{aligned}
r &= ce^{at}, \\
\psi &= bt + \theta.
\end{aligned}
\tag{8}
$$

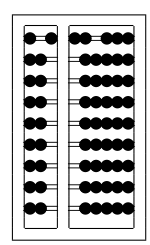
It follows the polar angle ψ moves in the course of time t uniformly with the velocity b. In other words, the ray, emanating from the origin and containing the point $x(t)$, rotates counterclockwise with the angular velocity b radian/second.

Excepts for the time t, we obtain the polar equation of the phase trajectory (8) in the form of a logarithmic spiral

$$
r = Ke^{\frac{a}{b}\psi}, \tag{9}
$$

where $K = ce^{-\frac{a}{b}\theta}$ is a positive constant, and the phase point moves counterclockwise along this trajectory. The phase portrait is an unstable focus (figure 2).

From (9) we deduce an imporant property of the phase trajectories: *Every two phase trajectories of the system (7) are obtained from each other by a homothety with the center at the origin and a positive radio.*



We now call our attention to the system (6) that differs from (7) by the presence of summands u^1, u^2.

For every point u of the plane R^2 we denote by v the point whose coor-

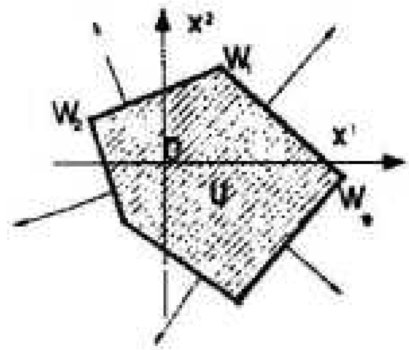


FIGURA 2.

dinates satisfy the relations

$$av^1 - bv^2 + u^1 = 0,$$
$$bv^1 + av^2 + u^2 = 0.$$

The point v is well-defined by u, since the linear system has nonzero determinant $a^2 + b^2$. The passing $u \underset{a}{\longrightarrow} v$ is a rotary dilation, *i.e.*, the composition of a rotation with the center at the origin and homothety with the center at the origin. We denote it by g, *i.e.*, $v = g(u)$. The rotation g maps the polygon U onto the polygon V with the vertices

$$v_1 = g(w_1), \dots, v_q = g(w_q).$$

Let $u(t)$, $t_0 \leq t \leq t_1$, be an optimal control for the controlled object (6), *i.e.*, $u(t)$ satisfies the maximum condition with respect to a nontrivial solution $\psi(t)$ of the conjugate system. Then, by the maximum principle, $u(t)$ is a piecewise constant and takes its values at the vertices of the polygon U. Let, in a time-segment, the optimal control $u(t)$ take the value $u = w_i$. Then in this time-segment, the phase point moves under the equations:

$$x^{1\prime} = ax^1 - bx^2 + w_i^1,$$
$$x^{2\prime} = bx^1 + ax^2 + w_i^2.$$

By the definition of the similarity g, this system can be rewritten in the form:

$$\frac{d}{dt}(x^1 - v_i^1) = a(x^1 - v_i^1) - b(x^2 - v_i^2), \qquad (10a)_i$$

$$\frac{d}{dt}(x^2 - v_i^2) = b(x^1 - v_i^1) + a(x^2 - v_i^2). \qquad (10b)_i$$

Thus if $u = w_i$ in a time-segment, then the corresponding piece of the trajectory is obtained from the corresponding piece of the trajectory for the homogeneous system (7) by the translation with the vector v_i.

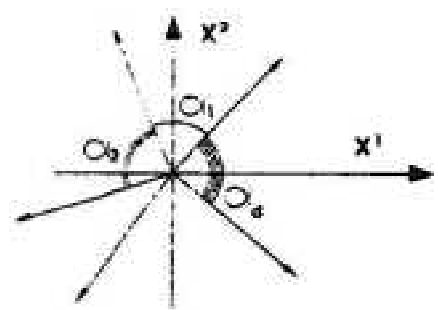


FIGURA 3.

We draw the rays emanating from the origin and having the directions defined by the outward normals of the polygon U (figure 3). Denote by α_i the angle between the rays which have the directions of outward normals for the sides adjoining the vertex w_i.

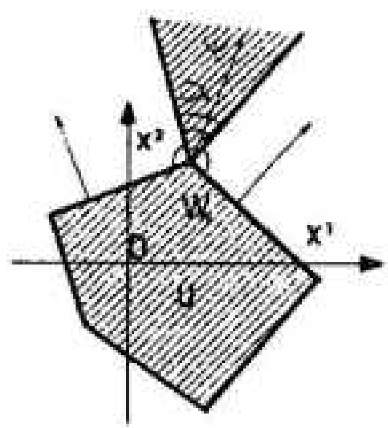


FIGURA 4.

Let now ψ be a nonzero vector situated in the interior of the angle α_i. Then ψ forms obtuse angles with both the sides of U adjoining the vertex w_i (figure 4). It follows that the scalar product $\langle \psi, u \rangle$ takes its maximal value over $u \in U$ at the vertex w_i.

The conjugate system for (6) has the form:

$$\psi_1' = -a\psi_1 - b\psi_2,$$
$$\psi_2' = b\psi_1 - a\psi_2.$$

Its general solution $\psi(t)$ has the following coordinate form:

$$\psi_1(t) = c'e^{-at}\cos(bt + \theta'),$$
$$\psi_2(t) = c'e^{-at}\sin(bt + \theta'),$$

where c', θ' are constants. Consequently the vector $\psi(t)$ rotates counterclockwise with the angular velocity b radian/second (a variation of its length is for us unessential). In other words, the vector $\psi(t)$ is changed in the following way. During the time α_i/b it is situated in the angle α_i, then during the time α_{i+1}/b it is situated in the angle α_{i+1}, then during the time α_{i+2}/b in the angle α_{i+2}, etc. We consider here the indices $i, i+1, i+2, \ldots$ as residues modulo q, *i.e.*, if for example $i = q$, then $i+1 = 1$, $i+2 = 2$, etcetera.

Now the view of optimal control $u(t)$ is clear: during the time α_i/b the control u takes he value w_i, then during the time α_{i+1}/b it takes the value w_{i+1}, then during the time α_{i+2}/b it takes the value w_{i+2}, etc. Finally the corresponding optimal trajectory (that which satisfies the maximum condition) has the following character: during the time $\leq \alpha_i/b$ the phase point moves under the system $(10)_i$, then during the time α_{i+1}/b it moves under the system $(10)_{i+1}$, then during the time α_{i+2}/b it moves under the system $(10)_{i+2}$ and so on; at last the phase point moves during the time $\leq \alpha_j/b$ under the system $(10)_j$ until it arrives at the origin. We put the sign $\leq$ for the first and the last pieces of the optimal trajectory, since the movement may start at a moment distinct from a switching and may be ended (by arriving to the origin) before the next switching.

We now remark that the arc of the trajectory for the system (7) running during the time α/b is visible from the origin at the angle of α (figure 5). This follows immediatly from the second equality (8).

Consequently the arc of the trajectory $(10)_i$ running during the time α/b is also visible from the point $v_i = g(w_i)$ at the angle of α.

Now every optimal trajectory may be described in the following way. The phase point moves along an arc of the system $(10)_i$ visible from v_i at an angle $\leq \alpha_i$; then the phase point describes an arc of $(10)_{i+1}$ visible from v_{i+1} at the angle α_{i+1}, then it describes an arc of $(10)_{i+2}$ visible from v_{i+2} at the angle α_{i+2}, and at last the phase point describes an arc of the system

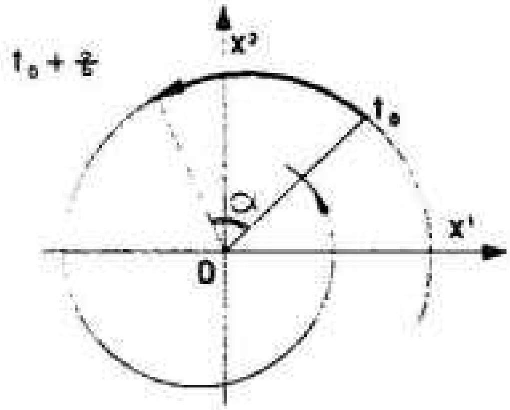


FIGURA 5.

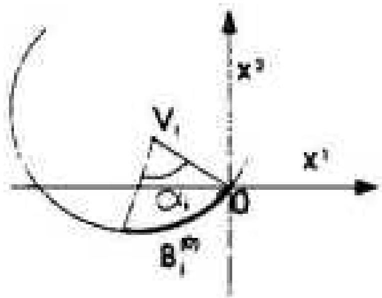


FIGURA 6.

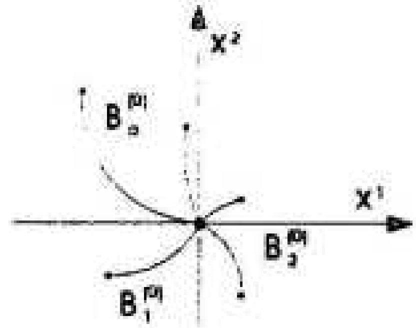


FIGURA 7.

$(10)_j$ visible from v_j at an angle $\leq \alpha_j$ and arrives at the origin. Conversely, every phase trajectory of this form is optimal.

We now are able to construct the synthesis of optimal trajectories. Denote by $B_j^{(0)}$ the arc of the system $(10)_j$ that terminates at the origin and is visible from v_j at the angle α_j, $j = 1, 2, \ldots, q$ (figure 6). We obtain q arcs $B_1^{(0)}, \ldots, B_q^{(0)}$ (figure 7). Let now $x(t)$ be an optimal trajectory and $u(t)$ be the corresponding optimal control. The last piece $x^{(0)}$ of the optimal trajectory $x(t)$ is situated in one of the arcs $B_j^{(0)}$, *i.e.*, this last piece is a part of $B_j^{(0)}$ from a point $a_j^{(0)} \in B_j^{(0)}$ till the origin. At the moment when $x(t)$ passes through the point $a_j^{(0)}$ the optimal control $u(t)$ has switching from $u = w_{j-1}$ to $u = w_j$. This means that before this moment the phase point $x(t)$ moved under the system $(10)_{j-1}$ during the time α_{j-1}/b. In other words, the previous piece $x^{(1)}$ of the optimal trajectory $x(t)$ is the arc of $(10)_{j-1}$ terminating at the point $a_j^{(0)}$ and visible from v_{j-1} at the angle α_{j-1}. We denote the initial point of this arc $x^{(1)}$ by $a_{j-1}^{(1)}$. As the point $a_j^{(0)}$ runs over $B_j^{(0)}$ the arcs $x^{(1)}$ fil in a "curvilinear quadrangle" (figure 8) whose two sides are $B_j^{(0)}, B_{j-1}^{(0)}$. We denote this quadrangle by $Q_{j-1}^{(1)}$ and its side oposite $B_j^{(0)}$ by $B_{j-1}^{(1)}$. The arc $B_{j-1}^{(1)}$ is the set of all points $a_{j-1}^{(1)}$ at which the switching from $(10)_{j-2}$ to $(10)_{j-1}$ takes place.

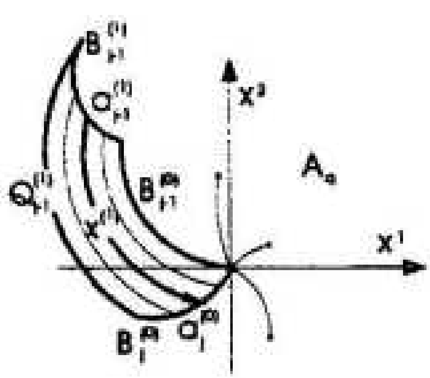


FIGURA 8.

Denote by h_i the homothety with the center v_i and ratio $e^{-a\alpha_i/b}$. By r_i denote the rotation at the angle α_i clockwise with the center v_i. The composition $p_i = h_i \operatorname{convol} r_i$ is a rotary dilition with fixpoint v_i. It is easily shown that the arc $B_{j-1}^{(1)}$ is obtained from $B_j^{(0)}$ by the transformation p_{j-1} (since, by (8), the point $a_{j-1}^{(1)}$ is obtained from $a_j^{(0)}$ under the transformation p_{j-1}.

Before the switching at the point $a_{j-1}^{(1)}$ the phase point $x(t)$ moved under the system $(10)_{j-2}$. The corresponding arc $x^{(2)}$ is visible from the point v_{j-2} at the angle α_{j-2} (figure 9). The initial point $a_{j-2}^{(2)}$ of this arc is obtained from $a_{j-1}^{(1)}$ by the transformation p_{j-2}. As the point $a_{j-1}^{(1)}$ runs over $B_{j-1}^{(1)}$ the arcs $x^{(2)}$ fill in a "curvilinear quadrangle". We denote this "quadrangle" by $Q_{j-2}^{(2)}$ and its side opposite $B_{j-1}^{(1)}$ by $B_{j-2}^{(2)}$. The arc $B_{j-2}^{(2)}$ is the set of all points $a_{j-2}^{(2)}$ at which the switching from $(10)_{j-3}$ to $(10)_{j-2}$ takes place and this arc $B_{j-2}^{(2)}$ is the image of $B_{j-1}^{(1)}$ under the transformation p_{j-2}.

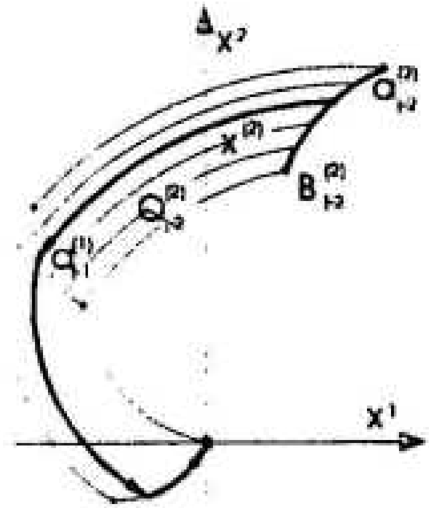


FIGURA 9.

Continuing in this way, we obtain q switching curves $F_1, \ldots, F_q$, where $F_i = B_i^{(0)} \cup B_i^{(1)} \cup B_i^{(2)} \cup \cdots$. The transformation p_i maps the curve F_{i+1} onto the curve $B_i^{(1)} \cup B_i^{(2)} \cup \cdots \subset F_i$. This allows to obtain successively the pieces of the switching curves $F_1, \ldots, F_q$ from their initial pieces $B_1^{(0)} \cdots B_q^{(0)}$.

The homothety h_i has ratio $e^{-a\alpha_i/b} < 1$. We put

$$k = \text{máx}\left(e^{-a\alpha_1/b}, \ldots, e^{-a\alpha_q/b}\right).$$

Then $0 < k < 1$. The arc $B_{i-1}^{(1)}$ is obtained from $B_i^{(0)}$ by the similarity $p_{i-1} = h_{i-1}\,\text{convol}\,r_{i-1}$ with radio $\leq k$. Similarly, for any $m = 1, 2, \ldots$, the arc $B_{j-1}^{(m+1)}$ is obtained from $B_j^{(m)}$ by the dilitation p_{j-1} with ratio $\leq k$. Hence in the sequence

$$B_i^{(0)}, B_{i-1}^{(1)}, B_{i-2}^{(2)}, \ldots, B_{i-m}^{(m)}, B_{i-m-1}^{(m+1)}, \ldots$$

every arc is obtained from the previous one by a similarity with ratio $\leq k$. In other words, the sizes of the arcs in this sequence decrease no slower than in geometrical progression with the ratio k. Consequently the controllability region, *i.e.*, the set $\Sigma \subset R^2$ in which the synthesis of optimal trajectories is realized, is bounded, open, and convex (figure 10).

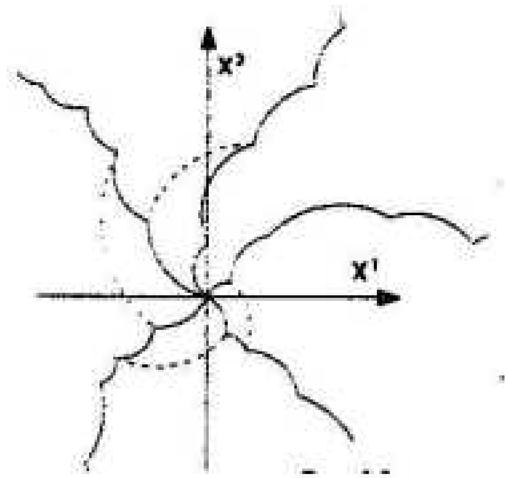


Figura 10.

We remark that the control parameter u takes value w_i in the "angle" between the curves F_i, and at the points of the curve F_i. This gives the synthesis of optimal controls. The synthesis of optimal trajectories is shown in figure 11.

Consider the curve composed from arcs $x^{(i-1)}, x^{(i-2)}, \ldots, x^{(i-q)}$. This curve starts at a point of the switching curve F_j (for an index j) and arr-

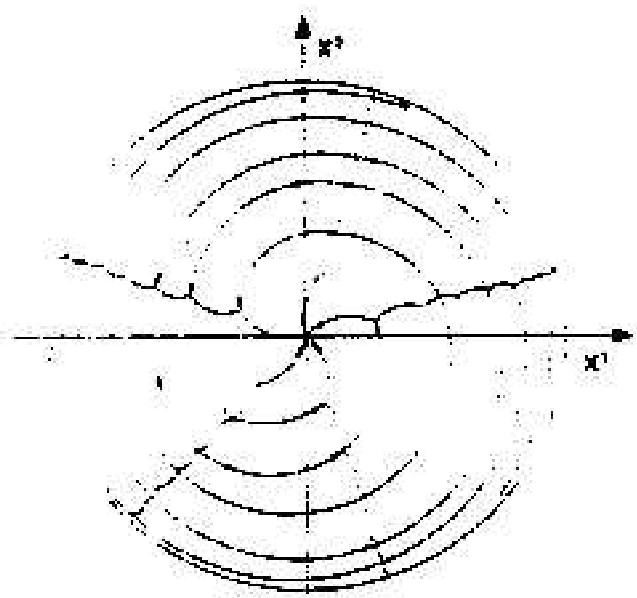


Figura 11.

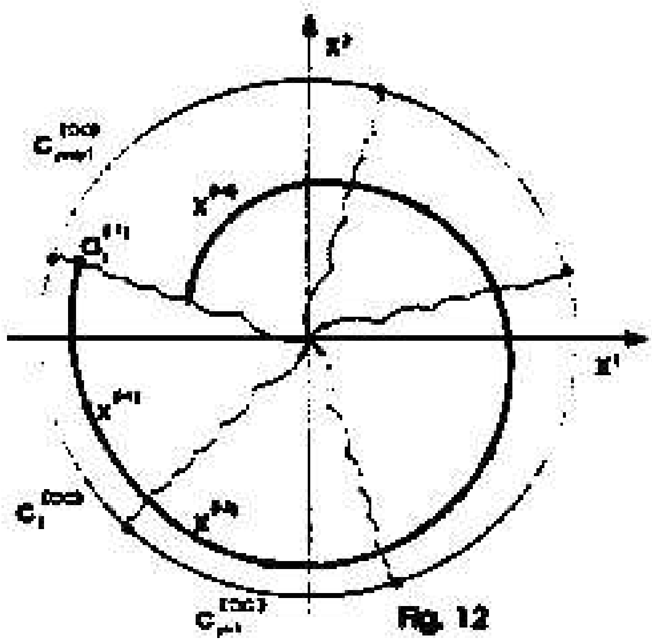


FIGURA 12.

ives to another point of the same switching curve F_j (figure 12). When the initial point $a_j^{(i-1)}$ of the arc $x^{(i-1)}$ tends to the boundary of Σ, the curve $x^{(i-1)} \cup x^{(i-2)} \cup \cdots \cup x^{(i-q)}$ tends to the curve $C_j^{(\infty)} \cup C_{j+1}^{(\infty)} \cup \cdots \cup C_{j+q-1}^{(\infty)}$ which runs over the boundary of the controllability region Σ. The arc $C_{j+k}^{(\infty)}$ is a piece of the trajectory for the equation $(10)_{j+k}$, *i.e.*, the phase point runs along the arc $C_{j+k}^{(\infty)}$ with $u(t) \equiv w_{j+k}$ during the time α_{j+k}/b, $k = 0, 1, \ldots, q-1$. This means that the cycle $\mathbf{bd}\Sigma = C_j^{(\infty)} \cup C_{j+1}^{(\infty)} \cup \cdots \cup C_{j+q-k}^{(\infty)}$ is a closed trajectory for the controlled object (6). In other words, if we take the initial point of the arc $C_j^{(\infty)}$ and consider the corresponding trajectory for the object (6) with $u = w_j$ during the time α_j/b, then the trajectory of $(10)_{j+1}$ during the time α_{j+1}/b, etc., then (after q switchings) we again arrive to the initial point of the arc $C_j^{(\infty)}$. This allows to find, uniquely, the cycle $\mathbf{bd}\Sigma = C_j^{(\infty)} \cup C_{j+1}^{(\infty)} \cup \cdots \cup C_{j+q-1}^{(\infty)}$.

The above reasoning shows that every optimal trajectory, which comes to the origin, approaches to the limit cycle $\mathbf{bd}\Sigma$ as $t \to -\infty$. Moreover, the limit cycle $\mathbf{bd}\Sigma$ is the closest trajectory satisfying the maximum principle.

In conclusion, we remark that the above results were considered for the controlled object (6). But, for any controlled object (1), that has complex eigenvalues with positive real parts, the synthesis of optimal trajectories is obtained from the synthesis for (6) by an affine transformation, *i.e.*, qualitatively the portrait of the synthesis is analogous.

III. Example

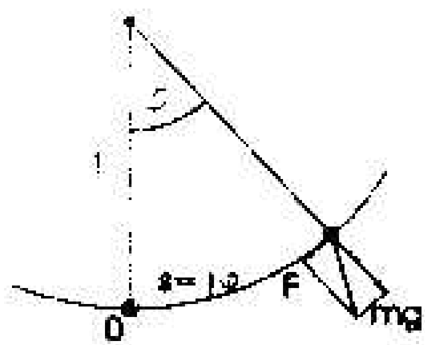


Figura 13.

Consider a mathematical pendulum with a negative, linear friction. We are interested in the movement of the pendulum close to its lowest equilibrium point O (figure 13). The equation of the movement has the form:

$$m\ddot{s} = ml\ddot{\psi} = -mg\sin\psi + b\dot{\psi},$$

where ψ is the deviation angle from the vertical direction, m is the mass of the pendulum, l is the length of the string, and $b > 0$ is the coefficient of the linear, negative friction. If we consider only small values of ψ, the equation takes the linearized form:

$$\ddot{\psi} - 2\delta\dot{\psi} + \omega^2\psi = 0, \tag{11}$$

where $\omega = \sqrt{q/l}$ and $\delta = b/2m$. Introducing the phase coordinates $z^1 = \psi$, $z^2 = \dot{z}^1 = w\,d\psi/dt$ (the angular velocity), we rewrite the movement equation in the form:

$$\dot{z}^1 = z^2,$$
$$\dot{z}^2 = -\omega^2 z^1 + 2\delta z^2.$$

By a suitable linear transformation of coefficients, this system can be reduced to the form (7).

The same equation (11) can be obtained, if we consider a mass point that moves under the action of a spring and a negative, linear friction.

Assume now that the mass point (or the pendulum) has two engines such that the homogeneous system (7) takes the form:

$$\begin{cases} \dot{x}^1 = ax^1 - bx^2 + u^1, \\ \dot{x}^2 = bx^1 + ax^2 + u^2, \\ -1 \le u^1 \le 1, \qquad -1 \le u^2 \le 1. \end{cases} \tag{12}$$

The system (12) is a particular case of the general system considered above. This allows to obtain the synthesis of optimal trajectories for these mechanical examples. Remark that the control region is a rectangle (*cf.* equation 12).

IV. The slowest going away trajectories

First of all, we reformulate the aforesaid results with the help of the Bellman function. For the linear controlled object (6) and a given initial point x_0 we denote by $\omega(x_0)$ the "minimal time" for transiting the point x_0 to the origin by an admissible control $u(t)$. The function $\omega(x_0)$ is said to be a *Bellman function.* If the control region U is a polygon in general position with respect to the matrix A, then the Bellman function $\omega(x)$ is defined and continuous in the open set $\Sigma \subset R^2$. Moreover, the Bellman function $\omega(x)$ is differentiable everywhere in $\Sigma \backslash (F_1 \cup \cdots \cup F_q)$ and satisfies the Bellman equation

$$\max_{u\in U}\Big(-\sum_{i=1}^{2}\frac{\partial\omega(x)}{\partial x^i}(Ax^i+u^i)\Big)=1,$$

i.e.,

$$\max_{u\in U}\langle -\operatorname{grad}\omega(x), Ax+u\rangle = 1.$$

and this maximun is obtained for every optimal process.

Moreover, for every admissible process $u(t)$, $x(t)$ ($t_0 \le t \le t_1$, with $x(t_0) = x_0$, $x(t_1) = x_1$ the inequality $t_1 - t_0 \ge \omega(x_1) - \omega(x_0)$ holds. This gives an estimate for the movement time.

We now generalize these facts for the trajectories situated in $R^2 \backslash \Sigma$. First we consider the object (6). Denote by L_i the ray emanating from the origin and passing through the common point of the arcs $C_{i-1}^{(\infty)}$ and $C_i^{(\infty)}$ (figure 14). Let $u(t), x(t)$ $(t_0 \leq t \leq t_1)$ be an admissible process satisfying the maximum principle and situated in $R^2 \backslash \Sigma$.

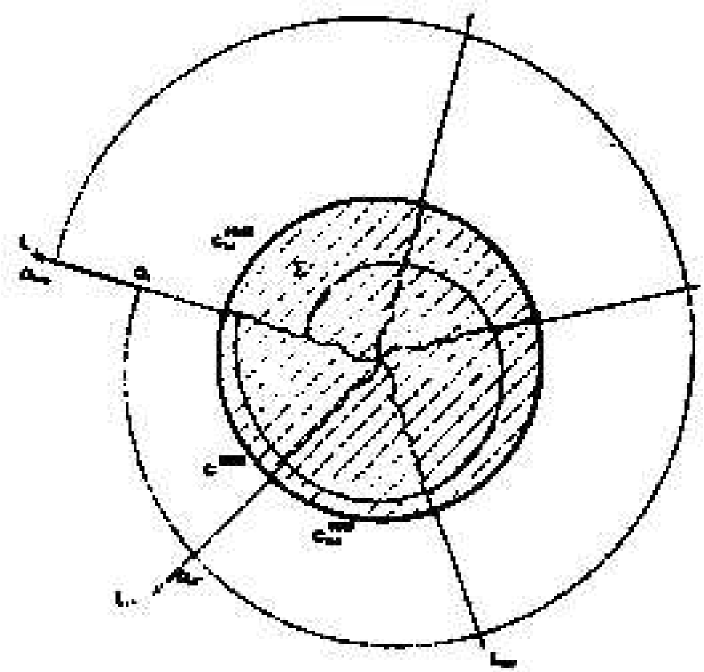


FIGURA 14.

Moreover, assume that the switching from w_{i-1} to w_i is realized as the point $x(t) = a_i$ is situated in the ray L_i. Then, during the time α_i/b the point $x(t)$ moves with $u \equiv w_i$ and is situated in the angle W_i between L_i and L_{i+1}. After that we have the switching from w_i to w_{i+1}, etcetera.

There exist a continuous function $\varpi(x)$ in $R^2 \backslash \Sigma$ that is diferentiable in the interior of every angle W_i. This function satisfies the equality $\varpi(x(t_1)) - \varpi(x(t_0)) = t_1 - t_0$ along the above trajectory $x(t)$ for which the maximum principle holds. On the segment $[a_i, a_{i+q}]$ the function $\varpi(x)$ increases from a value $\varpi(a_i) = \omega_i$ till the value $\varpi(a_{i+q}) = \omega_i + 2\pi/b$ (since the movement from a_i to a_{i+q} is realized during the time $2\pi/b$). And if the values of $\varpi(x)$ on the segment $[a_i, a_{i+q}]$ are fixed, then the function $\varpi(x)$ is uniquely defined everywhere on $R^2 \backslash \Sigma$ (along the trajectories satisfying the maximum principle).

Moreover, for every trajectory $x(t)$, $t_0 \leq t \leq t_1$, of the controlled object (6) situated in $R^2 \backslash \Sigma$, the estimate:

$$t_1 - t_0 \geq \overline{\omega}(x(t_1)) - \overline{\omega}(x(t_0)) \tag{14}$$

holds. This means that the trajectory $x(t)$ runs away most slowly if in (14) the equality holds. And this is realized if and only if the trajectory $x(t)$ with the control $u(t)$ satisfies the maximum condition. In other words, inside Σ, the maximum condition means the quickest approach to the origin, whereas outside Σ the maximum condition means the slowest running away. And the function $\overline{\omega}(x)$ satifies the relation (14) everywhere in $R^2 \backslash \Sigma$ except for the rays $L_1, \dots, L_q$.

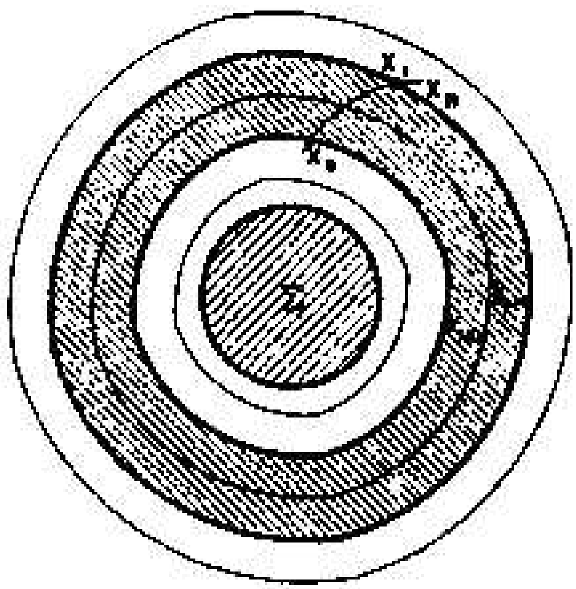


FIGURA 15.

The formulated facts may be explained in the following way. The curves $\overline{\omega}(x) = \text{const}$ form a system of closed curves around $\mathbf{bd}\Sigma$ (figure 15). We denote the curve $\overline{\omega}(x) = \mu$ by K_μ. Then for every trajectory $x(t)$ ($t_0 \leq t \leq t_1$), with $x(t_0) = x_0 \in K_{\mu_0}$, $x(t_1) = x_1 \in K_{\mu_1}$, we have $t_1 - t_0 \leq \mu_1 - \mu_0$, *i.e.*, the trajectory goes through the "ring" between K_{μ_0} and K_{μ_1} in a time $\leq \mu_1 - \mu_0$. For the trajectories which satisfy the maximum condition, the equality holds.

This means exactly that the trajectories satisfying the maximum condition are slowest in their running away. Figure 14 shows that all the trajectories outside Σ are like spirals going from $\mathbf{bd}\Sigma$ to infinity, and $\overline{\omega}(x(t))$ is the "away velocity" for them.

We remark that $\omega(x) \to -\infty$ as the point x approaches to $\mathbf{bd}\Sigma$, and $\omega(x) \to \infty$ as x goes to infinity. Moreover, the above description of the trajectories going away more slowly was given for the controlled object (6). Up to an affine transformation, the results are the same for any controlled object (1) that has complex eigenvalues with positive real parts.

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